Jan 2017

Math 511

Introduction

1.1 Problems leading to PDEs.

A) Deflection of a string of length L.

ρ: string's density per unit of length. M/L.

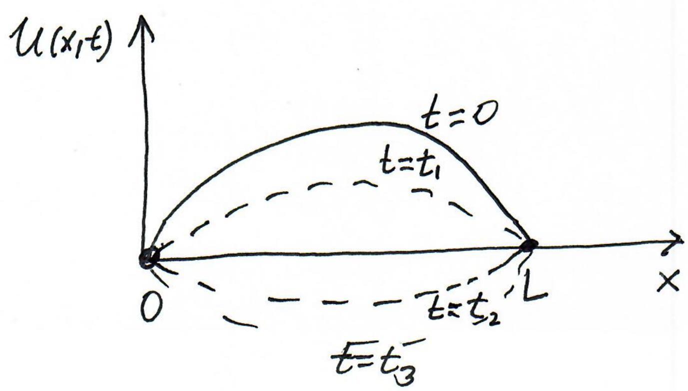
$u(x,t)$: position of differents points of string at time t. L

T: Tension in the string. Force $= M\frac{L}{t^2}$.

$u^0(x)$: initial deflection.

Initial boundary value Problem

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} = T\dfrac{\partial^2 u}{\partial x^2}, & 0<x<L,\ t>0 \\ u(x,0) = u^0(x), & 0<x<L. \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & 0<x<L \quad \text{(explain)} \\ u(0,t) = 0 & \\ u(L,t) = 0 & \text{(explain)} \end{cases}$$



Show MATLAB Simulation for a membrane.

Particular IBVP:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\
u(x,0) = \sin \frac{2\pi}{L} x, & 0 < x < L. \\
\dfrac{\partial u}{\partial t}(x,0) = 0 & \\
u(0,t) = 0, \quad u(L,t) = 0, & t > 0.
\end{cases}
$$

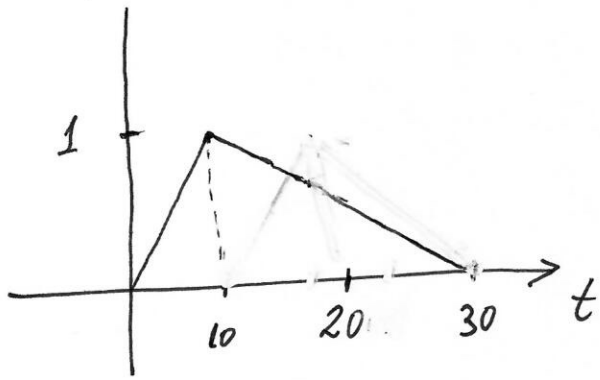
Soln: $u(x,t) = \cos\left(\frac{2\pi}{L} t\right) \sin\left(\frac{2\pi}{L} x\right)$

Homework: Write a MATLAB Code to show the vibrations of the string in this IBVP.

More general problem for wave equation 1D.

Vibrations of a String.

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 30, \quad t > 0. \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) = \begin{cases} x/10, & 0 \le x \le 10 \\ (30-x)/20, & 10 \le x \le 30 \end{cases} \\ u_t(x,0) = 0 \end{cases}$$



Soln:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}{}' \frac{9}{n^2\pi^2} \sin\left(\frac{n\pi}{3}\right) \sin \frac{n\pi}{30}x \cos \frac{2n\pi}{30}t}$$

Homework: Write MATLAB CODE to show Vibration of the string.

Use $\Delta t = 0.1$, $N_t = 600$, $\Delta x = L/N$, $N = 100$.

Consider $M = 30$ terms in the series.

Vibrations of a rectangular membrane

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \overset{=1}{c^2} \nabla^2 u, & \begin{matrix} 0<x<L \\ 0<y<H \end{matrix} \\
u(x,y,0) = \sin\left(\frac{2\pi}{L}x\right) \sin\left(\frac{\pi}{H}y\right) \\
\dfrac{\partial u}{\partial t}(x,y,0) = 0 \\
u(x,0,t) = 0, \quad u(x,H,t) = 0 \\
u(0,y,t) = 0, \quad u(L,y,t) = 0
\end{cases}
$$

Soln: $u(x,y,t) = \cos\left(\sqrt{\left(\frac{2\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2}\, t\right) *$

$$\sin\left(\frac{2\pi}{L}x\right) \sin\left(\frac{\pi}{H}y\right)$$

Show MATLAB simulation.

B) Conservation Laws

i) Thermodynamics 1st law

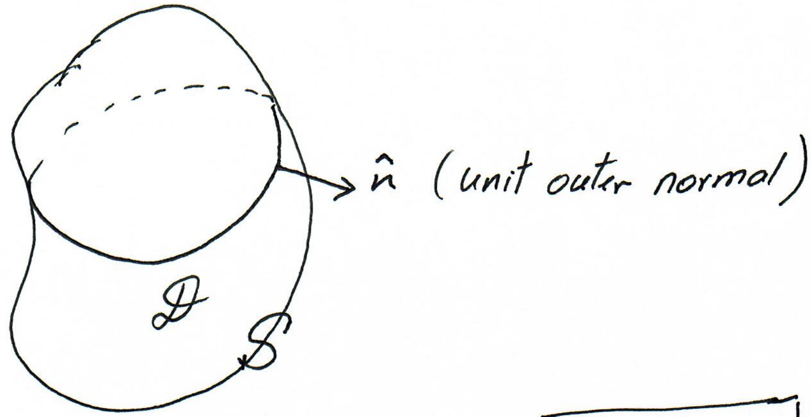
Change of internal energy = Heat added + Work done on system.

ii) Population biology

Rate of change of population = Birth rate − Death rate + Migration in and out of region.

Others : Mass, Momentum.

Domain $\mathcal{D}$ bounded by Surface S.



Rate of change of total amount of physical variable (APV) inside $\mathcal{D}$	=	Net rate of flow of APV across Surface S	+	Rate at which APV is added or absorbed inside domain $\mathcal{D}$

Construction of mathematical representation.

$\vec{u}(\vec{x},t)$: Density of physical variable (PV).

$\vec{\phi}(\vec{x},t)$: Flux of PV, $f(\vec{x},t)$: Sources or sinks of PV. per unit of Vol, per unit time.

Definition: Flux of PV is the amount of PV flowing through the domain bounding surface per unit of area and per unit ~~of~~ time.

Physical units: $[u] = \dfrac{APV}{L^3}$, $[\vec{\phi}] = \dfrac{APV}{t * L^2}$

$[f] = APV/_{t*L^3}$

$$\frac{d}{dt}\int_{\mathscr{D}} u(\vec{x},t)\,dv = -\int_{S} \vec{\phi}(\vec{x},t)\cdot\hat{n}(\vec{x})\,ds + \int_{\mathscr{D}} f(\vec{x},t)\,dv$$

If u and $\vec{\phi}$ are sufficiently smooth, e.g.,

$u, \vec{\phi} \in C'[\mathscr{D}\times \mathbb{R}^+]$. Then applying the divergence theorem

$$\int_{\mathscr{D}} \left[\frac{\partial u}{\partial t} + \nabla\cdot\vec{\phi} - f(\vec{x},t)\right](\vec{x},t)\,dv = 0 \qquad (5.1)$$

In general, the functions $u, \vec{\phi}$ and f are sufficiently smooth and $\mathscr{D}$ is arbitrary then (5.1) implies

$$\boxed{\frac{\partial u}{\partial t}(\vec{x},t) + \nabla\cdot\vec{\phi}(\vec{x},t) = f(\vec{x},t).} \qquad (5.2)$$

Equation representing a general conservation law

Smoothness condition to be satisfied by $u(\vec{x},t)$ and $\vec{\phi}(\vec{x},t)$ to transform (5.0) into (5.1)

Not the most general (or least restrictive).

I) **Leibniz's Rule**

$$\frac{d}{dt}\int_{\mathcal{D}} u(\vec{x},t)\,dv = \int_{\mathcal{D}} \frac{\partial u}{\partial t}(\vec{x},t)\,dv$$

Conditions: a) $u(\vec{x},t)$ is integrable with respect to the spatial variable $\vec{x}$ for t fixed.

b) $\frac{\partial u}{\partial t}(\vec{x},t)$ exists and is continuous on $(\mathcal{D}\cup S)\times[0,T]$.

Reference: Taylor - Mann Adv. Calc. pp. 582-583.

II)

$$\int_S \vec{\phi}\cdot\hat{n}\,ds = \int_{\mathcal{D}} \nabla\cdot\vec{\phi}\,dv$$

Conds: a) $\vec{\phi}$ is cont on $\mathcal{D}\cup S$.

b) $\vec{\phi}\in C'[\mathcal{D}]$, c) S is piecewise smooth.

Ref: Stewart, Taylor - Mann, Salas - Hille.

a) and b) can be expressed as

$\vec{\phi}\in C'[\mathcal{D}]\cap C[\mathcal{D}\cup S]$.

I) Conservation of mass

a) $u(\vec{x},t) = \rho(\vec{x},t)$: density $\quad [\rho] = \frac{M}{L^3}$

b) $\vec{\phi}(\vec{x},t)$: · mass flux $\quad [\vec{\phi}] = \frac{M}{t * L^2}$

Physically, mass is transported by particles of fluid flowing through the boundary surface S enclosing the volume $\mathcal{D}$.

This process is called <u>Convection</u>

Notice that

$$[\rho \vec{v}] = \frac{M}{L^2} \frac{L}{t} = \frac{M}{t * L^2}$$

has unit of flux. Thus, a good definition for the flux vector for transport of mass is given by

$$\vec{\phi}(\vec{x},t) = \rho \vec{v}.$$

c) $f(\vec{x},t)$: Sources or Sinks of mass $\quad [f] = \frac{M}{t * L^3}$

Substitution into (5.2) leads to

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = f \qquad (7.1)$$

If there are not Sources then $f \equiv 0$ and

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0}$$ Continuity equation

If the flow is Steady and incompressible

$$\rho(\vec{r}, t) \equiv \text{constant}$$

and (7.1) reduces to

$$\boxed{\nabla.\vec{v} = 0}$$

II) Heat Equation. Conservation of thermal energy

a) $u(\vec{x},t) = \rho\, c\, T(\vec{x},t)$: thermal energy density

where $\quad [u] = \dfrac{\text{Heat Energy}}{L^3}$

ρ: Mass density $\quad [\rho] = \dfrac{M}{L^3}$

c: specific heat $\quad [c] = \dfrac{\text{Energy}}{M \times \text{Temp}}$

$T(\vec{x},t)$: Temperature at $\vec{x}$ at time t.

b) $\vec{\phi}(\vec{x},t) = -K_0 \nabla T$: $\left\{\begin{array}{l}\text{Constitutive law} \\ \text{Known as Fourier's Law of} \\ \text{heat conduction.}\end{array}\right.$

$[\vec{\phi}] = \dfrac{\text{H. energy}}{t \ast L^2}$

Substitution into (5.2) assuming ρ, c, and K_0 are constants

$$\rho c \frac{\partial T}{\partial t} = K_0 \nabla \cdot (\nabla T) + f$$

where f represents Sources or Sinks of energy per unit of time and per unit of area.

If $f \equiv 0$, then

$$\rho c \frac{\partial T}{\partial t} = K_0 \nabla \cdot (\nabla t) = K_0 \nabla^2 T$$

$$\boxed{\frac{\partial T}{\partial t} = \kappa \nabla^2 T,} \qquad \frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \qquad (8.1)$$

where $\kappa \equiv \dfrac{K_0}{\rho c}$.

If steady conditions $\left(\frac{\partial}{\partial t} \equiv 0\right)$, then

$$\nabla^2 T = 0$$

The three equations:

1) $\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}$ or $\boxed{\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 u}{\partial x^2}}$ 1-dim wave equation.

2) $\boxed{\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}}$ 1-dim Heat equation.

3) $\nabla^2 T = 0$, or $\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$

Two-dimensional Laplace equation

are the Canonical forms of the three types of PDEs in the Classification of linear, second order PDEs in two independent variables.

Homework problem

The heat equation (8.1) (in page 8 of our notes) was derived using

$$\vec{\phi}(\vec{x},t) = -K_0 \nabla T$$

Conduction of thermal energy.

If we additionally assume that heat energy is transported by molecules moving at a constant velocity $\vec{V}$ the new flux is given by

$$\vec{\phi}(\vec{x},t) = -K_0 \nabla T + c\rho T \vec{V}$$

Convection of thermal energy

Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

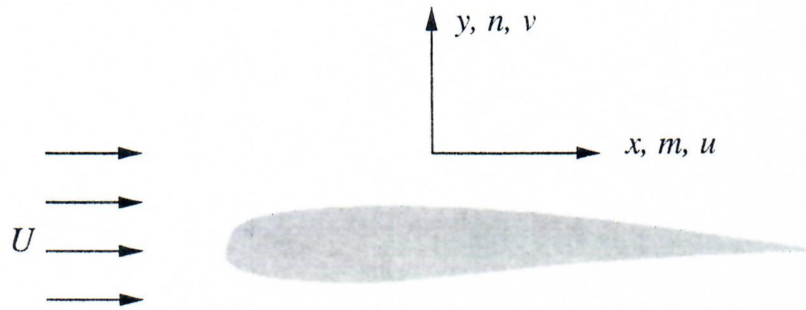


Figure 1.1.4: Typical flow region for Example 1.1.4.

Subscripts denote partial differentiation and ρ, m and n, e, and p are the fluid density, the x and y components of the momentum per unit volume, the total energy per unit volume, and the pressure, respectively. The x and y components of the velocity vector, u and v, are related to the components of the momentum vector by

$$m = \rho u, \qquad n = \rho v. \tag{1.1.5}$$

Finally, the pressure must be related to ρ, m, n, and e by an equation of state, which, for a perfect gas, is

$$p = (\gamma - 1)[e - (m^2 + n^2)/2\rho], \tag{1.1.6}$$

where γ is a constant. Equations (1.1.4a), (1.1.4b), (1.1.4c), and (1.1.4d) express the physical conditions that the mass, the x component of momentum, the y component of momentum, and the energy of the fluid, respectively, are conserved during its motion. A typical problem involving the flow around an airfoil is shown in Figure 1.1.4. The boundary conditions for this example would state that the momentum vector must be tangent to the airfoil and that flow conditions far from the airfoil are uniform and prescribed.

1.2 Second-Order Partial Differential Equations

Mathematicians classify partial differential systems according to the order of their highest derivatives, the number of independent variables, and the number of dependent variables. Other properties, such as linearity, are also used for characterization. Thus, Examples 1.1.1, 1.1.2, and 1.1.3 are second-order, linear, scalar problems in two variables while Example 1.1.4 is a first-order, nonlinear, vector system in the three variables x, y, and

t. We'll consider first-order problems in Section 1.3 and examine second-order problems here. In particular, we'll focus on a linear, scalar, partial differential equation in two variables having the form

$$\boxed{\mathcal{L}[u] \equiv au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g,} \tag{1.2.1}$$

where $a, b, \ldots, g$ are continuous functions of x and y and subscripts denote partial differentiation, *e.g.*, $u_x \equiv \partial u/\partial x$.

Such second-order equations are classified into three types depending on the roots

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - ac}}{a} \tag{1.2.2a}$$

of the characteristic equation

$$a\lambda^2 + 2b\lambda + c = 0. \tag{1.2.2b}$$

In particular, (1.2.1) is

- *hyperbolic* if λ_1, λ_2 are real and distinct, *i.e.*, if $b^2 - ac > 0$,
- *elliptic* if λ_1, λ_2 are complex, *i.e.*, if $b^2 - ac < 0$, and
- *parabolic* if $\lambda_1 = \lambda_2$, *i.e.*, if $b^2 - ac = 0$.

Example 1.2.1. A problem can be of a different type in different spatial regions. The equation

$$xu_{xx} + u_{yy} = 0, \tag{1.2.3}$$

has $a = x$, $b = 0$, and $c = 1$; thus, $\boxed{b^2 - ac = -x}$ and (1.2.3) is hyperbolic in the left-half plane ($x < 0$), elliptic in the right-half plane ($x > 0$), and parabolic on the line $x = 0$.

There are canonical partial differential equations of each type that are important for theoretical reasons and when developing practical numerical methods. The canonical hyperbolic equation is (1.2.1) with $a = 1$, $c = -1$, $b = d = e = f = g = 0$, or

$$u_{xx} - u_{yy} = 0. \tag{1.2.4}$$

hyperb. parab. elliptic

Consider the IBVP:

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ t > 0 & (13.1) \\ u(x,0) = \phi(x), & 0 \le x \le 1 & (13.2) \\ u(0,t) = u(1,t) = 0 & & (13.3) \end{cases}$$

A qualitative information of the behavior of the solution of (13) can be obtained by analyzing the L^2-norm of the soln without solving the problem:

In fact,

$$\|u(\cdot,t)\|_2 \equiv \left(\int_0^1 u^2(x,t)\,dx \right)^{1/2}$$

First multiply by u and integrate $\int_0^1 dx$

$$\int_0^1 (u u_t)(x,t)\,dx = \int_0^1 (u u_{xx})(x,t)\,dx$$

$$(u u_x)_x = u_x u_x + u u_{xx}$$

Assuming sufficient smoothness

$$\int_0^1 \frac{d}{dt}\left(\frac{u^2}{2}\right) dx = -\int_0^1 u_x^2\,dx + \int_0^1 (u u_x)_x\,dx$$

thus,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx = u u_x \Big|_0^1 - \int_0^1 u_x^2\,dx$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u(x,t)\|_2^2 = \overset{\overset{B.C.}{=0}}{u(1,t)}\,u_x(1,t) - \overset{\overset{B.C.}{=0}}{u(0,t)}\,u_x(0,t) - \int_0^1 u_x^2(x,t)\,dx$$

Then,

$$\frac{d}{dt}\|u(x,t)\|_2^2 = -\int_0^1 u_x^2(x,t)\,dx \le 0$$

By defining,

$$E(t) \equiv \int_0^1 u^2(x,t)\,dx = \|u(x,t)\|_2^2 \qquad (14.1)$$

leads to

$$\frac{dE}{dt}(t) \le 0 \qquad (14.2)$$

This implies that the function $E(t)$ is nonincreasing.

This analysis also serves to prove uniqueness of solutions for the BVP:

$$\begin{cases} u_t = u_{xx} & (15.1) \\ u(x,0) = \phi(x) & (15.2) \\ u(0,t) = h(t), \quad u(1,t) = g(t). & (15.3) \end{cases}$$

In fact, if $u_1(x,t)$ and $u_2(x,t)$ are solns. of

(15.1) - (15.3).

then $\boxed{w(x,t) = u_1(x,t) - u_2(x,t)}$ is also a

Solution of

$$\begin{cases} w_t = w_{xx} & (15.4) \\ w(x,0) = 0 & (15.5) \\ w(0,t) = 0, \quad w(1,t) = 0 & (15.6) \end{cases}$$

why?

And from previous analysis

$$\frac{d}{dt} E(t) = \frac{d}{dt} \int_0^1 w^2(x,t)\, dx \le 0 \qquad (15.7)$$

Also, for (15.4) - (15.6), $\quad E(0) = \int_0^1 w^2(x,0)\, dx = 0 \quad (15.8)$

and $E(t) = \int_0^1 w^2(x,t)\, dx \geq 0 \qquad (15.9)$

The results (15.7) - (15.9) plus the continuity of $E(t)$

leads to

$$E(t) \equiv 0$$

$$\int_0^1 w^2(x,t)\,dx = 0 \Rightarrow w(x,t) = 0$$

$$\Rightarrow \quad u_1(x,t) = u_2(x,t). \qquad (16.1)$$

(16.1) means that if there is soln. of (15.1)-(15.3) solution is unique.

and (1.2.12a) becomes

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8}{(k\pi)^2} e^{-k^2\pi^2 t} \sin k\pi x. \tag{1.2.13}$$

With this initial data, (1.2.13) is formally converging as $O(1/k^2)$. In general, the Fourier series converges as $O(1/k^{\sigma+2})$ if $\phi(x) \in C^\sigma$. Thus, even (some) discontinuous data leads to convergent Fourier series [3].

It will not be possible to obtain Fourier series or other explicit solutions to practical (variable-coefficient or nonlinear) problems. Nevertheless, we can obtain information about the solution without much effort. Let us, for example, obtain an estimate of the behavior of the solution in the $\mathcal{L}^2$ norm, which, for (1.2.10), is defined as

$$\|u(\cdot,t)\|_2 \equiv \left[\int_0^1 u(x,t)^2 dx\right]^{1/2}. \tag{1.2.14}$$

Multiplying (1.2.10a) by u and integrating on $(0,1)$, we find

$$\int_0^1 u u_t dx = \int_0^1 u u_{xx} dx.$$

integration by parts.

This may be written as

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx = \int_0^1 (u u_x)_x dx - \int_0^1 u_x^2 dx$$

or, using (1.2.14)

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_2^2 = u u_x|_0^1 - \int_0^1 u_x^2 dx.$$

Using the boundary conditions (1.2.10c)

$$\frac{d}{dt}\|u(\cdot,t)\|_2^2 = -2\int_0^1 u_x^2 dx.$$

Since the integral on the right is non-positive, the $\mathcal{L}^2$ norm of the solution is a decreasing function of t. The decrease is due to the presence of the u_{xx} term in (1.2.10a), which we call *dissipative*.

Problems

1. ([1], Section 10.7.) Find the Fourier-series solution of Laplace's equation

$$u_{xx} + u_{yy} = 0$$

on the rectangle $0 < x < a$, $0 < y < b$, satisfying the boundary conditions

$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad 0 < y < b,$$

$$u(x, 0) = 0, \qquad u(x, b) = g(x), \qquad 0 < x < a.$$

Additionally, find the solution in the particular case when

$$g(x) = \begin{cases} x, & 0 < x \le a/2 \\ a - x, & a/2 < x < a \end{cases}.$$

2. ([5], Section 8.) Find where the following partial differential equations are hyperbolic, parabolic, and elliptic:

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$x^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} + u = 0,$$

$$x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0.$$

3. Under suitable assumptions [2] the equations of two-dimensional, compressible, steady, inviscid flow ((1.1.4) with all time derivatives set to zero) can be reduced to the simpler system

$$(a^2 - u^2)u_x - uv(u_y + v_x) + (a^2 - v^2)v_y = 0,$$

$$v_x - u_y = 0.$$

Once again, u and v are the Cartesian coordinates of the velocity vector and a is the speed of sound

$$(a/a_0)^2 = 1 - \frac{\gamma - 1}{2}\frac{u^2 + v^2}{a_0^2},$$

with a_0 being the speed of sound when $u = v = 0$ and $\gamma > 1$ being a constant.

Introducing a potential function $\phi(x, y)$ such that

$$u = \phi_x, \qquad v = \phi_y$$

we satisfy the second partial differential equation and "reduce" the first to the *transonic full-potential equation*

$$(a^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} = 0.$$

Although this is a complicated nonlinear second-order differential equation, it can still be classified as hyperbolic, parabolic, or elliptic. However, with nonlinear equations, the type may depend on the unknown solution. Determine the type of the transonic full-potential equation. Express your answer in terms of the Mach number

$$M^2 = \frac{u^2 + v^2}{a^2} = \frac{\phi_x^2 + \phi_y^2}{a^2}.$$

4. Show that the $\mathcal{L}^2$ norm of the solution of Burgers' equation

$$u_t + uu_x = \sigma u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$

where σ is a positive constant, is a decreasing function of time when the initial and boundary conditions are

$$u(x,0) = \phi(x), \qquad 0 \le x \le 1,$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

1.3 Hyperbolic Conservation Laws: Characteristics, Shock Waves, and Rankine-Hugoniot Conditions

A conservation laws states that the total amount of some quantity remains unchanged during the evolution of the solution according to the partial differential equation. In physical processes without dissipation, these quantities might be the total mass, momentum, and energy. In this introductory chapter, let us confine our attention to conservation laws in one space dimension which typically have the form

$$\frac{d}{dt}\int_\alpha^\beta \mathbf{u}\,dx = -\mathbf{f}(\mathbf{u})|_\alpha^\beta = -\mathbf{f}(\mathbf{u}(\beta,t)) + \mathbf{f}(\mathbf{u}(\alpha,t)), \tag{1.3.1}$$